Partial fraction 2

1. Write $\frac{3-x}{(x^2+3)(x+3)}$ in partial fractions.

Put $\frac{3-x}{(x^2+3)(x+3)} = \frac{A}{x+3} + \frac{f(x)}{x^2+3}$

$A(x^2+3) + (x+3)f(x) = 3 - x \ldots (1)$

Put $x = -3$, Then $A = \left[\frac{3-x}{x^2+3}\right]_{x=-3} = \frac{3-(-3)}{(-3)^2+3} = \frac{1}{2}$

$\frac{1}{2}(x^2+3) + (x+3)f(x) = 3 - x$

$f(x) = \frac{(3-x)-\frac{1}{2}(x^2+3)}{x+3} = \frac{2(3-x)-(x^2+3)}{2(x+3)} = \frac{1-x}{2}$

$\therefore \frac{3-x}{(x^2+3)(x+3)} = \frac{1}{2(x+3)} + \frac{1-x}{2(x^2+3)}$

2. Partial fraction: $\frac{2R-4}{(R+1)(2R-1)(R-2)}$

$$\frac{2R-4}{(R+1)(2R-1)(R-2)} = \frac{2}{(R+1)(2R-1)} = \frac{\left[\frac{2}{R+1}\right]_{R=\frac{1}{2}}}{2R-1} + \frac{\left[\frac{2}{2R-1}\right]_{R=-1}}{R+1} = \frac{\frac{4}{3}}{\mathbf{2R-1}} - \frac{\frac{2}{3}}{\mathbf{R+1}}$$

3. Partial fraction: $\frac{x^3-x-5}{(x+2)\,(x^2+1)}$

Since both $x^3 - x - 5$ and $(x+2)\,(x^2+1)$ are monic and of degree three,

$\frac{x^3-x-5}{(x+2)\,(x^2+1)} \equiv 1 + \frac{A}{x+2} + \frac{Bx+C}{x^2+1} \ldots (1)$

$x^3 - x - 5 \equiv (x+2)(x^2+1) + A(x^2+1) + (Bx+C)(x+2) \ldots (2)$

Put $x = -2$ in (1), $(-2)^3 - (-2) - 5 \equiv A[(-2)^2 + 1]$

$\therefore A = -\frac{11}{5} \ldots (3)$

(3) ↓ (2), $x^3 - x - 5 \equiv (x+2)(x^2+1) - \frac{11}{5}(x^2+1) + (Bx+C)(x+2)$

$\therefore Bx + C = \frac{(x^3-x-5)-(x+2)(x^2+1)+\frac{11}{5}(x^2+1)}{x+2} = \frac{\frac{1}{5}(x^2-10x-24)}{x+2} = \frac{\frac{1}{5}(x+2)(x-12)}{x+2} = \frac{1}{5}(x-12)$

$\frac{x^3-x-5}{(x+2)\,(x^2+1)} \equiv \mathbf{1} - \frac{\frac{11}{5}}{\mathbf{x+2}} + \frac{\frac{1}{5}(\mathbf{x-12})}{\mathbf{x^2+1}}$

4. Partial fraction: $\frac{1}{(x+1)^3(x+2)}$ and deduce $\frac{1}{(x+1)^n(x+2)}$

Method 1

(1) $\frac{1}{(x+1)(x+2)} = \frac{(x+2)-(x+1)}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

(2) $\frac{1}{(x+1)^2(x+2)} = \frac{1}{x+1}\left[\frac{1}{(x+1)(x+2)}\right] = \frac{1}{x+1}\left[\frac{1}{x+1} - \frac{1}{x+2}\right] = \frac{1}{(x+1)^2} - \frac{1}{(x+1)(x+2)}$

$= \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+2}$

(3) $\frac{1}{(x+1)^3(x+2)} = \frac{1}{x+1}\left[\frac{1}{(x+1)^2(x+2)}\right] = \frac{1}{x+1}\left[\frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+2}\right]$

$= \frac{1}{(x+1)^3} - \frac{1}{(x+1)^2} + \frac{1}{(x+1)(x+2)} = \mathbf{\frac{1}{(x+1)^3} - \frac{1}{(x+1)^2} + \frac{1}{x+1} - \frac{1}{x+2}}$

Method 2

$\frac{1}{(x+1)^3(x+2)} + \frac{1}{x+2} = \frac{1}{x+2}\left[\left(\frac{1}{x+1}\right)^3 + 1\right] = \frac{1}{x+2}\left[\frac{1}{x+1} + 1\right]\left[\frac{1}{(x+1)^2} - \frac{1}{x+1} + 1\right]$

$= \frac{1}{x+2}\left[\frac{x+2}{x+1}\right]\left[\frac{1}{(x+1)^2} - \frac{1}{x+1} + 1\right] = \frac{1}{(x+1)^3} - \frac{1}{(x+1)^2} + \frac{1}{x+1}$

$\therefore \frac{1}{(x+1)^3(x+2)} = \mathbf{\frac{1}{(x+1)^3} - \frac{1}{(x+1)^2} + \frac{1}{x+1} - \frac{1}{x+2}}$

Method 3

$\frac{1}{(x+1)^3(x+2)} = \frac{[(x+1)^3+1]-(x+1)^3}{(x+1)^3(x+2)} = \frac{[(x+1)+1][(x+1)^2-(x+1)+1]-(x+1)^3}{(x+1)^3(x+2)}$

$= \frac{(x+2)[(x+1)^2-(x+1)+1]-(x+1)^3}{(x+1)^3(x+2)} = \mathbf{\frac{1}{x+1} - \frac{1}{(x+1)^2} + \frac{1}{(x+1)^3} - \frac{1}{x+2}}$

In general, $\mathbf{\frac{1}{(x+1)^n(x+2)} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(x+1)^{n-k+1}} - \frac{1}{x+2}}$

5. Evaluate $\sum_{n=1}^{N} \frac{2n^2+6n+6}{n(n+1)(n+2)(n+3)}$

$\sum_{n=1}^{N} \frac{2n^2+6n+6}{n(n+1)(n+2)(n+3)} = \sum_{n=1}^{N} \frac{n(n+1)+(n+2)(n+3)}{n(n+1)(n+2)(n+3)} = \sum_{n=1}^{N}\left[\frac{1}{(n+2)(n+3)} + \frac{1}{n(n+1)}\right]$

$= \sum_{n=1}^{N}\left[\frac{(n+3)-(n+2)}{(n+2)(n+3)}\right] + \sum_{n=1}^{N}\left[\frac{(n+1)-n}{n(n+1)}\right] = \sum_{n=1}^{N}\left[\frac{1}{n+2} - \frac{1}{n+3}\right] + \sum_{n=1}^{N}\left[\frac{1}{n} - \frac{1}{n+1}\right]$

$$= \left[\frac{1}{1+2} - \frac{1}{N+3}\right] + \left[\frac{1}{1} - \frac{1}{N+1}\right] = \frac{4}{3} - \frac{1}{N+1} - \frac{1}{N+3} = \mathbf{\frac{2N(2N+5)}{3(N+1)(N+3)}}$$

6. Partial fraction: $\frac{x^3}{(x+1)(x^2-1)^3}$

$$\frac{x^3}{(x+1)(x^2-1)^3} = \frac{x^3}{(x+1)^4(x-1)^3}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{1}{x+1} + \frac{1}{x-1}$$

Cube: $\frac{8x^3}{(x+1)^3(x-1)^3} = \frac{1}{(x+1)^3} + \frac{3}{(x+1)^2(x-1)} + \frac{3}{(x+1)}\frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}$

$$\frac{16x^3}{(x+1)^3(x-1)^3} = \frac{2}{(x+1)^3} + 3\frac{2}{(x+1)(x-1)}\left[\frac{1}{x+1} + \frac{1}{x-1}\right] + \frac{2}{(x-1)^3}$$

$$= \frac{2}{(x+1)^3} + 3\left[\frac{1}{x-1} - \frac{1}{x+1}\right]\left[\frac{1}{x+1} + \frac{1}{x-1}\right] + \frac{2}{(x-1)^3}$$

$$= \frac{2}{(x+1)^3} + \frac{3}{(x-1)^2} - \frac{3}{(x+1)^2} + \frac{2}{(x-1)^3}$$

Multiply both sides by $\frac{1}{x+1}$,

$$\frac{16x^3}{(x+1)^4(x-1)^3} = \frac{1}{x+1}\left[\frac{2}{(x+1)^3} + \frac{3}{(x-1)^2} - \frac{3}{(x+1)^2} + \frac{2}{(x-1)^3}\right]$$

$$= \frac{2}{(x+1)^4} - \frac{3}{(x+1)^3} + \frac{3}{(x+1)(x-1)^2} + \frac{2}{(x+1)(x-1)^3}$$

$$= \frac{2}{(x+1)^4} - \frac{3}{(x+1)^3} + \frac{3}{(x+1)(x-1)^2} + \frac{2}{(x+1)(x-1)}\left[\frac{1}{(x-1)^2}\right]$$

$$= \frac{2}{(x+1)^4} - \frac{3}{(x+1)^3} + \frac{3}{(x+1)(x-1)^2} + \left[\frac{1}{x-1} - \frac{1}{x+1}\right]\left[\frac{1}{(x-1)^2}\right]$$

$$= \frac{2}{(x+1)^4} - \frac{3}{(x+1)^3} + \frac{2}{(x+1)(x-1)^2} + \frac{1}{(x-1)^3}$$

$$= \frac{2}{(x+1)^4} - \frac{3}{(x+1)^3} + \left[\frac{1}{x-1} - \frac{1}{x+1}\right]\left[\frac{1}{x-1}\right] + \frac{1}{(x-1)^3}$$

$$= \frac{2}{(x+1)^4} - \frac{3}{(x+1)^3} + \frac{1}{(x-1)^2} - \frac{1}{(x+1)(x-1)} + \frac{1}{(x-1)^3}$$

$$\frac{32x^3}{(x+1)^4(x-1)^3} = \frac{4}{(x+1)^4} - \frac{6}{(x+1)^3} + \frac{2}{(x-1)^2} - \frac{2}{(x+1)(x-1)} + \frac{2}{(x-1)^3}$$

$$= \frac{4}{(x+1)^4} - \frac{6}{(x+1)^3} + \frac{2}{(x-1)^2} - \left[\frac{1}{x-1} - \frac{1}{x+1}\right] + \frac{2}{(x-1)^3}$$

$$\frac{x^3}{(x+1)^4(x-1)^3} = \mathbf{\frac{1}{32}\left[\frac{4}{(x+1)^4} - \frac{6}{(x+1)^3} + \frac{1}{x+1} + \frac{2}{(x-1)^3} + \frac{2}{(x-1)^2} - \frac{1}{x-1}\right]}$$

7. Partial fraction: $\frac{1}{(x-1)^2(x+1)}$

$\frac{1}{(x-1)^2(x+1)}$

$= \frac{1}{2}\left[\frac{(x+1)-(x-1)}{(x-1)^2(x+1)}\right] = \frac{1}{2}\left[\frac{1}{(x-1)^2}\right] - \frac{1}{2}\left[\frac{1}{(x-1)(x+1)}\right] = \frac{1}{2}\left[\frac{1}{(x-1)^2}\right] - \frac{1}{4}\left[\frac{(x+1)-(x-1)}{(x-1)(x+1)}\right]$

$= \frac{\frac{1}{2}}{(x-1)^2} - \frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{x+1}$

8. Partial fraction: $\frac{(2s^3-s^2)}{(4s^2-4s+5)^2}$

$E = \frac{(2s^3-s^2)}{(4s^2-4s+5)^2} = \frac{(8s^3-4s^2)}{4(4s^2-4s+5)^2}$

Using long division, divide $(8s^3 - 4s^2)$ by $(4s^2 - 4s + 5)$, we can get

the quotient $= 2s + 1$ and remainder $= -6s - 5$.

$\therefore 8s^3 - 4s^2 = (4s^2 - 4s + 5)(2s + 1) - 6s - 5$

Hence $E = \frac{(4s^2-4s+5)(2s+1)-6s-5}{4(4s^2-4s+5)^2} = \frac{2s+1}{4s^2-4s+5} + \frac{-6s-5}{4(4s^2-4s+5)^2}$

9. Partial fraction: $\frac{x^2+1}{(x-1)^2(x+1)}$

$\frac{x^2+1}{(x-1)^2(x+1)} = \frac{(x-1)(x+1)+2}{(x-1)^2(x+1)} = \frac{1}{x-1} + \frac{2}{(x-1)^2(x+1)} = \frac{1}{x-1} + \frac{(x+1)-(x-1)}{(x-1)^2(x+1)}$

$= \frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{1}{(x+1)(x-1)} = \frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right] = \frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{(x-1)^2} + \frac{1}{2}\left(\frac{1}{x-1}\right)$

10. Partial fraction: $\frac{1}{x(x^2+1)^2}$

This can be done just by inspection. First observe that:

$\frac{1}{x(x^2+1)} = \frac{(x^2+1)-x^2}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$ ... (1)

$\frac{1}{x(x^2+1)^2} = \frac{1}{x^2+1}\left[\frac{1}{x(x^2+1)}\right] = \frac{1}{x^2+1}\left[\frac{1}{x} - \frac{x}{x^2+1}\right]$, by (1)

$= \frac{1}{x(x^2+1)} - \frac{x}{(x^2+1)^2}$

$= \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}$, by (1)

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